

Pretty good state transfer on double stars

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Abstract

Let A be the adjacency matrix of a graph X and suppose $U(t) = \exp(itA)$. We view A as acting on $\mathbb{C}^{V(X)}$ and take the standard basis of this space to be the vectors e_u for u in $V(X)$. Physicists say that we have *perfect state transfer* from vertex u to v at time τ if there is a scalar γ such that

$$U(\tau)e_u = \gamma e_v.$$

(Since $U(t)$ is unitary, $\|\gamma\| = 1$.) For example, if X is the d -cube and u and v are at distance d then we have perfect state transfer from u to v at time $\pi/2$. Despite the existence of this nice family, it has become clear that perfect state transfer is rare. Hence we consider a relaxation: we say that we have *pretty good* state transfer from u to v if there is a complex number γ and, for each positive real ϵ there is a time t such that

$$\|U(t)e_u - \gamma e_v\| < \epsilon.$$

Again we necessarily have $|\gamma| = 1$.

In Godsil, Kirkland, Severini and Smith [7] it is shown that we have pretty good state transfer between the end vertices of the path P_n if and only $n + 1$ is a power of two, a prime, or twice a prime.

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(There is perfect state transfer between the end vertices only for P_2 and P_3 .) It is something of a surprise that the occurrence of pretty good state transfer is characterized by a number-theoretic condition.

In this paper we extend the theory of pretty good state transfer. We provide what is only the second family of graphs where pretty good state transfer occurs. The graphs we use are the *double-star* graphs $S_{k,\ell}$, these are trees with a vertex of degree $k+1$ adjacent to a vertex of degree $\ell+1$, and all other vertices of degree one. We prove that perfect state transfer does not occur in any graph in this family. We show that if $\ell > 2$, then there is pretty good state transfer in $S_{2,\ell}$ between the two end vertices adjacent to the vertex of degree three. If $k, \ell > 2$, we prove that there is never perfect state transfer between the two vertices of degree at least three, and we show that there is pretty good state transfer between them if and only these vertices both have degree $k+1$ and $4k+1$ is not a perfect square. Thus we find again the the existence of perfect state transfer depends on a number theoretic condition. It is also interesting that although no double stars have perfect state transfer, there are some that admit pretty good state transfer.

1 Introduction

Let X be a graph on n vertices with adjacency matrix A and let $U(t)$ denote the matrix-valued function $\exp(iAt)$. We note that $U(t)$ is both symmetric and unitary, and that it determines what is called a *continuous quantum walk*. Work in quantum computing has raised many questions about the relation between physically interesting properties of $U(t)$ and properties of the graph X . For recent surveys see [4], [8].

The physical properties of interest to us in this paper are *perfect state transfer* and *pretty good state transfer*. Assume $n = |V(X)|$ and identify the coordinates of \mathbb{R}^n with $V(X)$. If $u \in V(X)$, we use e_u to denote the standard basis vector indexed by u . If u and v are distinct vertices of X we say we have *perfect state transfer* from u to v at time τ if there is a complex number γ such that

$$U(\tau)e_u = \gamma e_v.$$

Since $U(t)$ is unitary, $|\gamma| = 1$. The evidence is that perfect state transfer is uncommon, and we consider a relaxation of it. We say that we have *pretty good state transfer* from u to v if there is a complex number γ and, for each

positive real ϵ there is a time t such that

$$\|U(t)e_u - \gamma e_v\| < \epsilon.$$

Again we necessarily have $|\gamma| = 1$. Pretty good state transfer was introduced in [4].

There is a considerable literature on perfect state transfer. In a seminal paper on the topic Christandl et al. [3] show that there is perfect state transfer between the end vertices of the paths P_2 and P_3 , but perfect state transfer does not occur between the end vertices of any path on four or more vertices. From [4] we know that, for any integer k , there are only finitely many connected graphs with maximum valency at most k on which perfect state transfer occurs.

Much less is known about pretty good state transfer. In [4] it is shown that it takes place on P_4 and P_5 . Vinet and Zhedanov [12] have studied pretty good state transfer on weighted paths with loops Godsil, Kirkman, Severini and Smith that pretty good state transfer occurs between the end-vertices of P_n if and only $n + 1 = 2^m$, or if $n + 1 = p$ or $2p$ where p is an odd prime. It is surprising to see that the existence of pretty good state transfer depends so delicately on the prime divisors of $n + 1$.

In this paper we provide a second class of graphs where pretty good state transfer occurs if and only if a number theoretic condition holds. Let $S_{k,k}$ denote the graph we get by taking two copies of $K_{1,k}$ and joining the two vertices of degree k by a new edge. We show that there is pretty good state transfer between the vertices of degree $k + 1$ in $S_{k,k}$ if and only if $4k + 1$ is not a perfect square. We also show that there is never perfect state transfer between these two vertices. We conclude the paper with some remarks that show that if pretty good state transfer does occur on a graph, then, in a sense, it must occur regularly.

2 Quotients

We introduce a useful tool. A more expansive treatment will be found in [6, Ch. 9].

Let X be a graph. A partition π of vertex set $V(X)$ with cells

$$C_1, C_2, \dots, C_r$$

is *equitable*, if the number of neighbors in C_j of any vertex u in C_i is a constant b_{ij} . The directed graph with the r cells of π as its vertices and b_{ij} arcs from the i -th to the j -th cells of π is called the *quotient* of X over π , and denoted by X/π . The entries of the adjacency matrix of this quotient graph are given by $A(X/\pi)_{i,j} = b_{ij}$. We can symmetrize $A(X/\pi)$ to B by letting $B_{i,j} = \sqrt{b_{ij}b_{ji}}$. We call the (weighted) graph with adjacency matrix B the *symmetrized quotient graph*. In the following we always use B to denote the symmetrized form of the matrix $A(X/\pi)$.

If π is a partition of $V(X)$, its characteristic matrix, denoted by P , is the 01-matrix whose columns are the characteristic vectors of the cells of π , viewed as subsets of $V(X)$. If we normalize the characteristic matrix P such that each column have length one, then we obtain the *normalized characteristic matrix* of π , denoted by Q . Note that $Q^T Q = I$ and $Q Q^T$ is a block diagonal matrix with diagonal blocks $\frac{1}{r} J_r$, where J_r is the all-ones matrix of order $r \times r$, and the size of the i -th block is the size of the i -th cell of π . The vertex u forms a singleton cell of π if and only if $Q Q^T e_u = e_u$.

For the sake of convenience, in the following text, we always denote by $\{u\}$ the singleton cell $\{u\}$.

Since A is symmetric, it has a spectral decomposition

$$A = \sum_r \theta_r E_r$$

where θ_r runs over the distinct eigenvalues θ_r of A and E_r is the matrix that represents orthogonal projection onto the eigenspace belonging to θ_r .

2.1 Lemma. *Let π be an equitable partition of X with normalized characteristic matrix Q . Let A be the adjacency matrix of X and let B be the adjacency matrix of the symmetrized quotient graph. Then the idempotents in the spectral decomposition of B are the non-zero matrices $Q^T E_r Q$, where E_r runs over the idempotents in the spectral decomposition of A .*

Proof. As π is equitable, $AQ = QB$. Hence $A^k Q = QB^k$ and so if $f(t)$ is a polynomial then $f(A)Q = Qf(B)$. There is a polynomial f_r such that $f_r(A) = E_r$, and hence

$$E_r Q = Q f_r(B).$$

Then $f_r(B) = Q^T E_r Q$ is symmetric, we show it is idempotent. We have

$$(Q^T E_r Q)^2 = Q^T E_r Q Q^T E_r Q$$

and since QQ^T commutes with A , it commutes with E_r . Since $Q^T Q = I$ we then have

$$Q^T E_r Q Q^T E_r Q = Q^T Q Q^T E_r Q = Q^T E_r Q.$$

It follows that

$$B = Q^T A Q = \sum_r \theta_r Q^T E_r Q \quad (2.1)$$

and since

$$\sum_r Q^T E_r Q = Q^T Q = I$$

we conclude that (2.1) is the spectral decomposition of B . \square

If $\{u\}$ and $\{v\}$ are singletons in π , the 2×2 submatrix of $Q^T E_r Q$ indexed by $\{u\}$ and $\{v\}$ is equal to the 2×2 submatrix of E_r indexed by u and v . Since u and v are strongly cospectral if and only if for each idempotent E_r , we have $(E_r)_{u,u} = (E_r)_{v,v} = \pm (E_r)_{u,v}$.

In [1], R. Bachman et al. studied perfect state transfer of quantum walks on quotient graphs. We state their result without proof. For the details, see [1, Theorem 2].

2.2 Lemma. *Let X be a graph with an equitable partition π and assume $\{a\}$ and $\{b\}$ are singleton cells of π . Let B denote the adjacency matrix of the symmetrized quotient graph relative to π . Then, for any time t ,*

$$(e^{-itAx})_{a,b} = (e^{-itB})_{\{a\},\{b\}}$$

and therefore G has perfect state transfer from a to b at time t if and only if the symmetrized quotient graph has perfect state transfer from $\{a\}$ to $\{b\}$. \square

3 Strongly Cospectral Vertices

Let u and v be vertices in X . We say that u and v are *cospectral vertices* if the characteristic polynomials $\phi(X \setminus u, t)$ and $\phi(X \setminus v, t)$ are equal. If $\theta_1, \dots, \theta_m$ are the distinct eigenvalues of X and the matrices E_1, \dots, E_m are the orthogonal projections onto the corresponding eigenspaces we have the spectral decomposition

$$A = \sum_r \theta_r E_r.$$

From [4] we know that u and v are cospectral if and only if $(E_r)_{u,u} = (E_r)_{v,v}$, for all r . Since $(E_r)_{u,u} = \|E_r e_u\|^2$, we see that u and v are cospectral if and only if the projections $E_r e_u$ and $E_r e_v$ have the same length for each r . We say that u and v are *strongly cospectral* if, for each r ,

$$E_r e_v = \pm E_r e_u.$$

[9] observed that if there is perfect state transfer from vertices u to v , then u and v are strongly cospectral. In [4], an argument due to Dave Witte is presented, which shows that if there is pretty good state transfer from vertex u to vertex v , then u and v are strongly cospectral. If the eigenvalues of A are simple, then two vertices are strongly cospectral if and only if they are cospectral.

3.1 Lemma. *Vertices u and v are strongly cospectral if and only if for each idempotent E_r , we have $(E_r)_{u,u} = (E_r)_{v,v} = \pm(E_r)_{u,v}$.*

Proof. The vertices u and v are cospectral if and only if $(E_r)_{u,u} = (E_r)_{v,v}$ for each r . Set $y = E_r e_u$ and $z = E_r e_v$. By Cauchy-Schwarz

$$0 \leq \|y\|^2 \|z\|^2 - |y^T z|^2$$

and equality holds if and only if $\{y, z\}$ is linearly dependent. Since

$$\|y\|^2 \|z\|^2 - |y^T z|^2 = (E_r)_{u,u} (E_r)_{v,v} - (E_r)_{u,v}^2,$$

and since $\|E_r e_u\| = \|E_r e_v\|$ when u and v are cospectral, the lemma follows. \square

We will show that for the symmetric double star graphs $S_{k,k}$, the two central vertices are strongly cospectral.

We now consider the behaviour of strongly cospectral vertices under quotients over an equitable partition.

3.2 Lemma. *Let X be a graph and let π be an equitable partition of X in which $\{u\}$ and $\{v\}$ are singleton cells. Then u and v are strongly cospectral in X if and only if $\{u\}$ and $\{v\}$ are strongly cospectral in the symmetrized quotient graph X/π .*

Proof. Let Q be the normalized characteristic matrix of π and let B be the symmetrized quotient matrix. If E_r is an idempotent in the spectral decomposition of A , then $E_r = p_r(A)$ for some polynomial p and so

$$E_r Q = p_r(A) Q = Q p_r(B).$$

Therefore $p_r(B) = Q^T E_r Q$ and

$$A E_r Q = \theta_r E_r Q = \theta_r A Q p_r(B) = \theta_r Q B p_r(B),$$

from which it follows that the non-zero matrices $Q^T E_r Q$ are the idempotents in the spectral decomposition of B . If $a, b \in \{u, v\}$, then since $\{u\}$ and $\{v\}$ are singleton cells of π , we have

$$(Q^T E_r Q)_{a,b} = (E_r)_{a,b}.$$

We conclude that u and v are strongly cospectral in X if and only if they are strongly cospectral in X/π . \square

4 Perfect State Transfer

Let S_k and S_ℓ be the two star graphs with k and ℓ edges respectively. Then the double star graph, which we denote by $S_{k,\ell}$, is the graph obtained by joining the two vertices with degrees k and ℓ of S_k and S_ℓ , respectively. We call a double star graph a *symmetric double star* if $k = \ell$.

In this section we will show that double star graphs do not have perfect state transfer

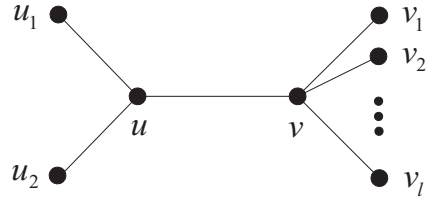


Figure 1: The double star graph $S_{2,\ell}$

4.1 Lemma. *Let $S_{2,\ell}$ be a double star graph. Let π be the equitable partition with cells*

$$\{\{\{u_1\}, \{u_2\}, \{u\}, \{v\}, N(v) \setminus \{u\}\}.$$

If $S_{2,\ell}$ has perfect state transfer between u_1 and u_2 , then $\exp(i\theta_i t) = -1$ for each nonzero eigenvalue θ_i .

Proof. Suppose u and v are the two central vertices of $S_{2,\ell}$, that u_1, u_2 are the two neighbors of u and v_1, \dots, v_ℓ are the neighbors of v . Then π is an equitable partition with cells

$$\{\{u_1\}, \{u_2\}, \{u\}, \{v\}, N(v) \setminus \{u\}\}$$

and

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{\ell} \\ 0 & 0 & 0 & \sqrt{\ell} & 0 \end{pmatrix}$$

is the adjacency matrix of the corresponding symmetrized quotient graph. We have perfect state transfer between u_1 and u_2 if and only if the symmetrized quotient graph has perfect state transfer between vertex $\{u_1\}$ and $\{u_2\}$. Let

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the symmetrized quotient graph have perfect state transfer between vertex $\{u_1\}$ and $\{u_2\}$ if and only $U_B(t) = \exp(iBt) = \gamma F$, where $\|\gamma\| = 1$. The eigenvalues of B are

$$\begin{aligned} \theta_1 &= 0, \\ \theta_2 &= \frac{1}{2}\sqrt{2\ell + 6 + 2\sqrt{\ell^2 - 2\ell + 9}}, \\ \theta_3 &= -\frac{1}{2}\sqrt{2\ell + 6 + 2\sqrt{\ell^2 - 2\ell + 9}}, \\ \theta_4 &= \frac{1}{2}\sqrt{2\ell + 6 - 2\sqrt{\ell^2 - 2\ell + 9}}, \\ \theta_5 &= -\frac{1}{2}\sqrt{2\ell + 6 - 2\sqrt{\ell^2 - 2\ell + 9}}. \end{aligned}$$

Let E_i be the idempotent of θ_i , where $i = 1, \dots, 5$. The symmetrized quotient graph has perfect state transfer between vertex $\{u_1\}$ and $\{u_2\}$ if and only if

$$U_B(t) = \sum_i \exp(i\theta_i t) E_i = \gamma F, \quad (4.1)$$

where $\|\gamma\| = 1$. Note that $(-1, 1, 0, 0, 0)$ is an eigenvector for θ_1 and

$$E_1 = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This implies that $I - 2E_1 = F$. Note that $\sum_i E_i = I$ and

$$F = -E_1 + E_2 + E_3 + E_4 + E_5.$$

This, together with (4.1) and the fact $\exp(i\theta_1 t) = 1$, implies that $\gamma = -1$ and $\exp(i\theta_i t) = -1$ for $i = 2, 3, 4, 5$. \square

4.2 Lemma. *There is no perfect state transfer between u_1 and u_2 in the double star graph $S_{2,\ell}$.*

Proof. By Lemma 4.1, $S_{2,\ell}$ has perfect state transfer between u_1 and u_2 if and only if $\exp(i\theta_i t) = -1$ for $\theta_i \neq 0$, where θ_i is as given in the proof of Lemma 4.1. Hence $\theta_i t = (2k_i + 1)\pi$ for some $k_i \in \mathbb{Z}$. In particular,

$$\frac{\theta_2}{\theta_4} = \frac{2k_2 + 1}{2k_4 + 1} \tag{4.2}$$

The right hand of this equation is rational. Next we show that the left hand is irrational.

$$\begin{aligned} \frac{\theta_2}{\theta_4} &= \sqrt{\frac{2\ell + 6 + 2\sqrt{\ell^2 - 2\ell + 9}}{2\ell + 6 - 2\sqrt{\ell^2 - 2\ell + 9}}} \\ &= \frac{\sqrt{2\ell}(\ell + 3 + \sqrt{\ell^2 - 2\ell + 9})}{4\ell} \end{aligned}$$

Note first that

$$\ell^2 - 2\ell + 9 = (\ell - 1)^2 + 8$$

If a and b are integers and with $a < b$ and $b^2 - a^2$ is even, then $b - a$ is even. As $(a + 2k)^2 - a^2 = 4a(a + k)$, we see that $\ell^2 - 2\ell + 9$ is a perfect square if and only if $\ell = 2$. Hence if $\ell \neq 2$ then θ_2/θ_4 is irrational and thus (4.2)

cannot hold. This means that we cannot have perfect state transfer from u_1 to u_2 when $\ell \neq 2$.

If $\ell = 2$, then $\theta_2 = 2$, $\theta_4 = 1$ and

$$\frac{2k_2 + 1}{2k_4 + 1} = 2,$$

which is impossible. Therefore, we cannot have perfect state transfer between u_1 to u_2 on the double star graph $S_{2,\ell}$. \square

Next, we study the state transfer between two central vertices u and v on the double star graph $S_{k,k}$.

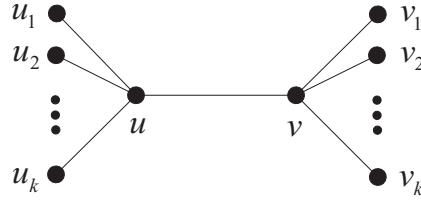


Figure 2: The double star graph $S_{k,k}$

4.3 Lemma. *Let X be the double star graph $S_{k,k}$. Then*

$$\exp(iAt)_{u,v} = ((1 - 2\beta) \sin \alpha t + 2\beta \sin(1 - \alpha)t)i.$$

where $\alpha = \frac{1+\sqrt{1+4k}}{2}$ and $\beta = \frac{k}{1+4k+\sqrt{1+4k}}$.

Proof. Let π be an equitable partition with cells $\{N(u) \setminus v, \{u\}, \{v\}, N(v) \setminus u\}$. Then the adjacency matrix of the quotient graph X/π is

$$A_{X/\pi} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

After symmetrizing, we get

$$B = \begin{pmatrix} 0 & \sqrt{k} & 0 & 0 \\ \sqrt{k} & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{k} \\ 0 & 0 & \sqrt{k} & 0 \end{pmatrix}$$

The eigenvalues of B are $\alpha, 1 - \alpha, \alpha - 1, -\alpha$. The corresponding eigenvectors are the columns of the following matrix:

$$\begin{pmatrix} -1 & -1 & 1 & 1 \\ -\frac{\sqrt{k}}{\alpha} & \frac{\sqrt{k}}{\alpha-1} & \frac{\sqrt{k}}{\alpha-1} & -\frac{\sqrt{k}}{\alpha} \\ \frac{\sqrt{k}}{\alpha} & -\frac{\sqrt{k}}{\alpha-1} & \frac{\sqrt{k}}{\alpha-1} & \frac{\sqrt{k}}{\alpha} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence

$$\begin{aligned} \exp(iBt)_{\{u\},\{v\}} &= \left(\frac{1}{2} - \beta\right) \exp(i\alpha t) + \beta \exp(i(1 - \alpha)t) \\ &\quad - \beta \exp(-i(1 - \alpha)t) - \left(\frac{1}{2} - \beta\right) \exp(-i\alpha t) \\ &= ((1 - 2\beta) \sin(\alpha t) + 2\beta \sin((1 - \alpha)t))i. \end{aligned}$$

By Lemma 2.2, we have

$$\exp(iAt)_{u,v} = \exp(iBt)_{\{u\},\{v\}}.$$

The result follows. \square

4.4 Lemma. *Let $S_{k,k}$ be a double star graph. Then there is no perfect state transfer from one vertex of degree $k + 1$ to the other.*

Proof. Note that $2\beta = \frac{2k}{1+4k+\sqrt{1+4k}}$ and so we have $0 < \beta < 1$. Hence

$$\begin{aligned} \|\exp(iAt)_{u,v}\| &= |(1 - 2\beta) \sin(\alpha t) + 2\beta \sin((1 - \alpha)t)| \\ &\leq |1 - 2\beta| + |2\beta| \\ &= 1. \end{aligned}$$

Equality holds if and only if $\sin \alpha t = \sin(1 - \alpha)t = \pm 1$. Without loss of generality, assume that $\sin \alpha t = \sin(1 - \alpha)t = 1$. Then

$$\alpha t = \frac{\pi}{2} + 2m\pi, \tag{4.3}$$

$$(1 - \alpha)t = \frac{\pi}{2} + 2n\pi. \tag{4.4}$$

It follows that $\alpha = \frac{4m+1}{4(m+n)+2}$. This implies that α is not an integer.

On the other hand, suppose $\delta = 1 + 4k$ and $\alpha = \frac{1+\sqrt{\delta}}{2}$. Then we can rewrite equations (4.3) and (4.4) in the following form:

$$\begin{aligned}\frac{1 + \sqrt{\delta}}{2}t &= \frac{\pi}{2} + 2m\pi, \\ \frac{1 - \sqrt{\delta}}{2}t &= \frac{\pi}{2} + 2n\pi\end{aligned}$$

and hence

$$\frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}} = \frac{1 + 4m}{1 + 4n} \in \mathbb{Q}.$$

Note that on the other hand,

$$\frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}} = \frac{1 + \delta + 2\sqrt{\delta}}{1 - \delta},$$

which implies that δ is a perfect square. Since $\delta = 1 + 4k$ is odd, we can assume that $\delta = (2s + 1)^2$, thus $\alpha = \frac{1+\sqrt{\delta}}{2} = s + 1$ is an integer. Contradiction. \square

Note that if G is a group of automorphisms of the graph X and $u \in V(X)$, then G_u denotes the subgroup consisting of the automorphisms that fix u . We have following result from [4].

4.5 Lemma. *Let G be the automorphism group of X . If we have perfect state transfer from vertex u to vertex v in X , then $G_u = G_v$.* \square

Next, we will give out main result in this section.

4.6 Theorem. *There is no perfect state transfer on double star graph $S_{k,l}$.*

Proof. Suppose u and v are the two central vertices of $S_{k,\ell}$, that u_1, \dots, u_k are the neighbors of u and v_1, \dots, v_ℓ are the neighbors of v . By Lemma 4.5, if $k, \ell \geq 3$, perfect state transfer can only occur between the two central vertices. Now if we have perfect state transfer between two vertices of X , they must be cospectral and therefore must have the same degree. So $k = \ell$ and perfect state transfer can only occur between two central vertices u and v . By Lemma 4.4, there is no perfect state transfer between two central vertices u and v in $S_{k,k}$.

Suppose then that k and ℓ are not both at least three, then the graph is either the path P_4 or $S_{2,\ell}$. We know that there is no perfect state transfer on P_4 in [2]

For $S_{2,\ell}$, we consider two cases. First, if $\ell \neq 2$, then by Lemmas 3.1 and 4.5, the only vertices which might have perfect state transfer are vertices with degree one that are adjacent to the vertex with degree three. By Lemma 4.2, we know there is no perfect state transfer in this case.

If $\ell = 2$, then perfect state transfer might occur between two vertices of degree one with a common neighbor, or between the two vertices of degree three. By Lemma 4.2 and Lemma 4.4, we also do not have perfect state transfer in this case. \square

5 Pretty Good State Transfer

In this section, we will investigate pretty good state transfer on double star graph. We say we have pretty good state transfer from u to v if there is a sequence $\{t_k\}$ of real numbers and a scalar γ such that

$$\lim_{k \rightarrow \infty} U(t_k) = \gamma e_v,$$

where $\|\gamma\| = 1$.

5.1 Lemma. *Let X be a graph and π is an equitable partition with $\{u\}$ and $\{v\}$ are singletons. Then there is pretty good state transfer from vertex u to vertex v if and only if there is pretty good state transfer from $\{u\}$ to $\{v\}$ in the symmetrized quotient graph with adjacency matrix B .*

Proof. By Lemma 2.2, we have $\exp(iBt_k)_{\{u\},\{v\}} = \exp(iAt_k)_{u,v}$. Hence

$$\lim_{t \rightarrow \infty} \exp(iAt_k)_{u,v} = \gamma$$

if and only if

$$\lim_{t \rightarrow \infty} \exp(iBt_k)_{\{u\},\{v\}} = \gamma. \quad \square$$

5.2 Theorem. *Let $S_{2,\ell}$ be a double star graph, u_1 and u_2 are two pendent vertices adjacent to the vertex with degree three. Then $S_{2,\ell}$ has pretty good state transfer from vertex u_1 to u_2 if and only if $\ell \neq 2$.*

Proof. If $\ell = 2$, then the eigenvalues of $S_{2,2}$ are integers. So in this case $S_{2,2}$ is periodic and since we know perfect state transfer does not occur, pretty good state transfer does not occur either.

Suppose then that $\ell \neq 2$. By Lemma 4.1, we see that there is pretty good state transfer from u_1 to u_2 if and only if there is a sequence of times $(t_k)_{k \geq 0}$ such that

$$\lim_{k \rightarrow \infty} \exp(i\theta_i t_k) = -1 \quad \text{for } i = 2, 3, 4, 5.$$

Note that if $\lim_{k \rightarrow \infty} \exp(i\theta_2 t_k) = -1$, then $\lim_{k \rightarrow \infty} \exp(i\theta_3 t_k) = -1$. Similarly, if $\lim_{k \rightarrow \infty} \exp(i\theta_4 t_k) = -1$, then $\lim_{k \rightarrow \infty} \exp(i\theta_5 t_k) = -1$. The question becomes whether we can choose integers m, n such that

$$\frac{\theta_2}{\theta_4} \approx \frac{2m+1}{2n+1},$$

that is, whether we can choose integers m, n such that

$$m\theta_4 + n(-\theta_2) \approx \frac{1}{2}(\theta_2 - \theta_4)$$

If θ_2 and θ_4 are linearly independent over \mathbb{Q} , then by Kronecker's approximation theorem the set

$$\{m\theta_4 - n\theta_2 : m, n \in \mathbb{Z}\}$$

is dense in \mathbb{R} . Therefore we can choose $m, n \in \mathbb{Z}$, such that

$$\frac{\theta_2}{\theta_4} \approx \frac{2m+1}{2n+1}$$

This implies we can choose a series $\{t_k\}_{k \geq 0}$ such that $\lim_{k \rightarrow \infty} \exp(i\theta_i t_k) = -1$ for all $i = 2, 3, 4, 5$.

Next, we show that θ_2 and θ_4 are linearly independent over \mathbb{Q} . Assume that for some $x, y \in \mathbb{Q}$,

$$x\theta_2 + y\theta_4 = 0$$

Then

$$\frac{\theta_2}{\theta_4} = -\frac{y}{x}$$

is rational. However, by the proof of Lemma 4.2, we know θ_1/θ_2 is irrational. Contradiction. \square

Next, we will investigate the pretty good state transfer on double star $S_{k,k}$.

5.3 Theorem. *Let $S_{k,k}$ be a symmetric double star graph. Then either*

- (a) *$1+4k$ is not a perfect square and there is pretty good state transfer from vertex u to vertex v , or*
- (b) *$1+4k$ is a perfect square and there is not pretty good state transfer from vertex u to vertex v .*

Proof. If $4k+1$ is a perfect square then the eigenvalues of $S_{k,k}$ are rational and hence integers. So in this case $S_{k,k}$ is periodic and since we know that perfect state transfer does not occur, pretty good state transfer does not occur either.

Assume then that $4k+1$ is not a perfect square. By Lemma 4.3 we see that there is pretty good state transfer from u to v if and only if there is a sequence of times $(t_\ell)_{\ell \geq 0}$ such that

$$\lim_{\ell \rightarrow \infty} \sin \alpha t_\ell = \lim_{\ell \rightarrow \infty} \sin(1-\alpha)t_\ell = \pm 1. \quad (5.1)$$

Note that if $\lim_{\ell \rightarrow \infty} \sin \alpha t_\ell = \pm 1$ then $\lim_{\ell \rightarrow \infty} \cos \alpha t_\ell = 0$. Since

$$\cos t_\ell = \cos \alpha t_\ell \cos(1-\alpha)t_\ell - \sin \alpha t_\ell \sin(1-\alpha)t_\ell$$

we see that if (5.1) holds then $\lim_{\ell \rightarrow \infty} \cos t_\ell = -1$.

This implies that $t_\ell \approx (2m+1)\pi$ and $\alpha t_\ell \approx n\pi + \frac{\pi}{2}$ for $m, n \in \mathbb{Z}$. The question becomes whether we can choose integers m, n such that $(2m+1)\alpha - n \approx \frac{1}{2}$.

As $1+4k$ is not a perfect square, α is an irrational number. So α and $\frac{1}{2}$ are linearly independent over the rationals, and hence by Kronecker's approximation theorem the set

$$\{m\alpha - n : m, n \in \mathbb{Z}\}$$

is dense in \mathbb{R} . Therefore we can choose $m, n \in \mathbb{Z}$, such that

$$m\alpha - n \approx \frac{1}{4} - \frac{1}{2}\alpha.$$

This implies we can choose a series $\{t_\ell\}$ such that both $\lim_{\ell \rightarrow \infty} \sin \alpha t_\ell = \pm 1$ and $\lim_{\ell \rightarrow \infty} \cos t_\ell = -1$. \square

6 Recurrence

If A is the adjacency matrix of a graph X then the set

$$\{U(t) : t \in \mathbb{R}\}$$

is an abelian group, which we denote by G . In fact G is a 1-parameter subgroup of the unitary group on \mathbb{C}^n , where $n = |V(X)|$, and so its closure \overline{G} is an abelian Lie group. Since G is connected, so is its closure and therefore \overline{G} is isomorphic to a direct product of some number of copies of \mathbb{R}/\mathbb{Z} . Asking whether there is perfect state transfer from u to v is equivalent to asking whether there is a matrix M in G such that

$$M_{u,u} = M_{v,v} = 0, \quad M_{u,v} = M_{v,u} = \gamma \quad (6.1)$$

where $|\gamma| = 1$. Asking whether there is pretty good state transfer is asking whether there is a matrix M in \overline{G} such that these conditions hold.

If we prove that pretty good state transfer does occur, we have shown that there is a sequence of times t_ℓ such that $t_\ell \rightarrow \infty$ and, for each $\epsilon > 0$ there is time t_ℓ such that $U(t_\ell)$ is within ϵ of a solution to the conditions of (6.1). However we can say something more concrete, using the following.

6.1 Lemma. *If $\epsilon > 0$, there is a time T such that each element of \overline{G} lies within ϵ of an element of $\{U(s+t) : 0 \leq t \leq T\}$, for any s .*

Proof. Define

$$S = \{M \in \overline{G} : \|M - I\| < \epsilon\}$$

Then S is open and its translates under the action of G cover \overline{G} . Since \overline{G} is compact, some finite set of translates of S cover \overline{G} . Hence there is a time T such that all elements of \overline{G} lie within ϵ of an element of $\{U(t) : 0 \leq t \leq T\}$. Since $U(t)$ is unitary it follows that each element of \overline{G} lies within ϵ of an element of

$$\{U(t+s) : 0 \leq t \leq T\},$$

for any s . We conclude that any element of \overline{G} lies within ϵ of an element of $\{U(t+s) : 0 \leq t \leq T\}$, for any s . \square

6.2 Corollary. *If we have pretty good state transfer from u to v in X then for each positive ϵ there is a real number T such that, for each s , the set $\{U(t+s) : 0 \leq t \leq T\}$ contains an element within ϵ of a solution to (6.1).*

Since $U(0) = I$, the arguments above also yield the conclusion that, if $\epsilon > 0$ then there is a time T such that in each real interval of length T there is a time t such that $\|U(t) - I\| < \epsilon$. Thus we can say that any graph is *approximately periodic*. (We recall that a graph is *periodic* if there is a time T such that $U(t) = \gamma I$, for some complex number γ with norm 1. Periodic graphs are studied, and characterized, in [5]; their eigenvalues must be square roots of integers, and consequently these graphs are rare.)

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